The NURBS and GeoPDEs packages
Octave software for research on IGA

Rafael Vázquez

IMATI 'Enrico Magenes', Pavia
Consiglio Nazionale delle Ricerche

Motivation

In 2008/2009 in Pavia we started to work in isogeometric analysis within the GeoPDEs project. Software development was one of the objectives.

Starting point: different codes, different problems, different developers.

First goal: a uniform implementation of the different codes.

Second goal: it should be clear and easy to use, for didactic purposes, and for new researchers coming into the research group.

The result were two Octave packages: the **NURBS** package, for geometry construction and manipulation, and **GeoPDEs**, for isogeometric methods.

1. **The NURBS package**: B-splines and NURBS
    - B-splines and NURBS: mathematical definitions
    - Functions and examples
2. **The GeoPDEs package**: isogeometric analysis
    - Isogeometric analysis: definition
    - The development of GeoPDEs
    - Some examples

1. **The NURBS package**: B-splines and NURBS
    - B-splines and NURBS: mathematical definitions
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    - Isogeometric analysis: definition
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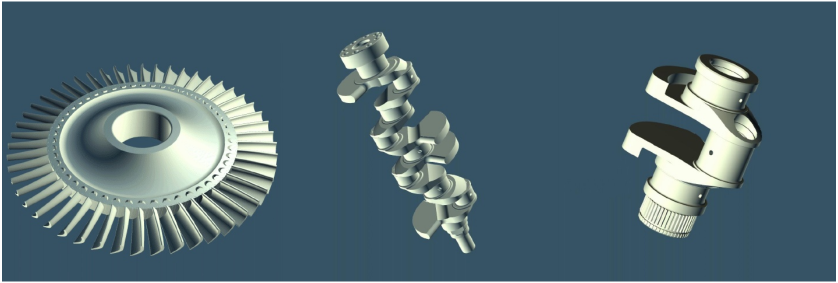
Non Uniform Rational B-Splines (NURBS)

NURBS (non-uniform rational B-splines) are probably the most commonly used CAD technology for engineering design.



NURBS are a generalization of **B-splines**.

B-splines: definition

Given an ordered knot vector $\xi_1 \leq \ldots \leq \xi_{n+p+1}$,

define the n B-splines of degree p by the recursion formula

$$N_{i,0}(\zeta) = \begin{cases} 1 & \text{if } \xi_i \leq \zeta \leq \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,p}(\zeta) = \frac{\zeta - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\zeta) + \frac{\xi_{i+p+1} - \zeta}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\zeta)$$

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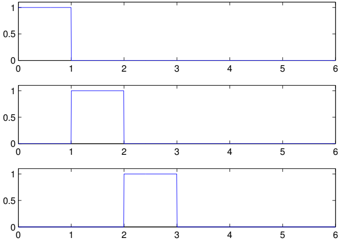
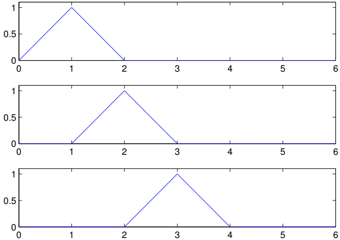


Degree 0



Degree 1

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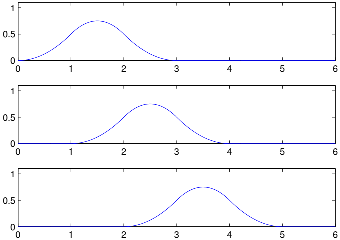
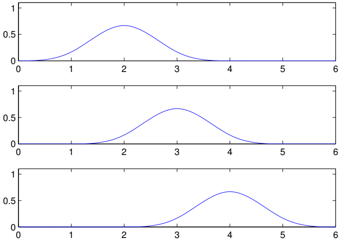
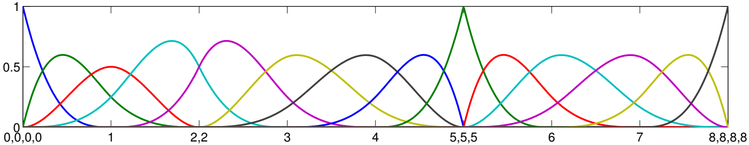


Degree 2



Degree 3

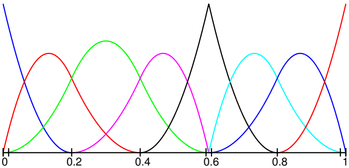
B-splines: the basis functions

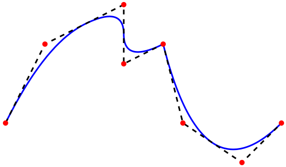


B-spline **basis functions** have the following properties:

- They are non-negative and form a partition of unity.
- Locally linearly independent on each knot span (ξ_i, ξ_{i+1})
- The function $N_{i,p}$ is supported in the interval $[\xi_i, \xi_{i+p+1}]$.
- Piecewise polynomials of degree p, and regularity at most $p-1$.
- The **regularity** at ξ_i is controlled by the **knot multiplicity**.

B-spline curves: definition





A B-spline curve in $\mathbb{R}^d$ is defined as a linear combination of B-splines:

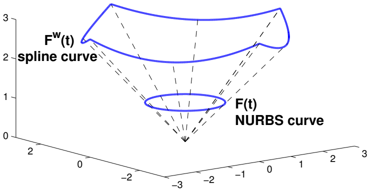
$$\mathbf{F}(\zeta) = \sum_{i=1}^{n} \mathbf{C}_i N_{i,p}(\zeta)$$

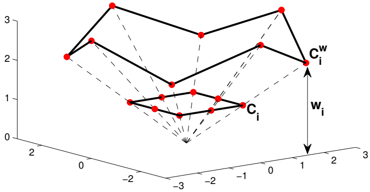
To define the parametrization $\mathbf{F}$ we only need:

- The basis functions $N_{i,p}$, given by the knot vector.
- The control points $\mathbf{C}_i \in \mathbb{R}^d$.

NURBS curves: definition

NURBS are rational B-splines, used to represent conic sections.





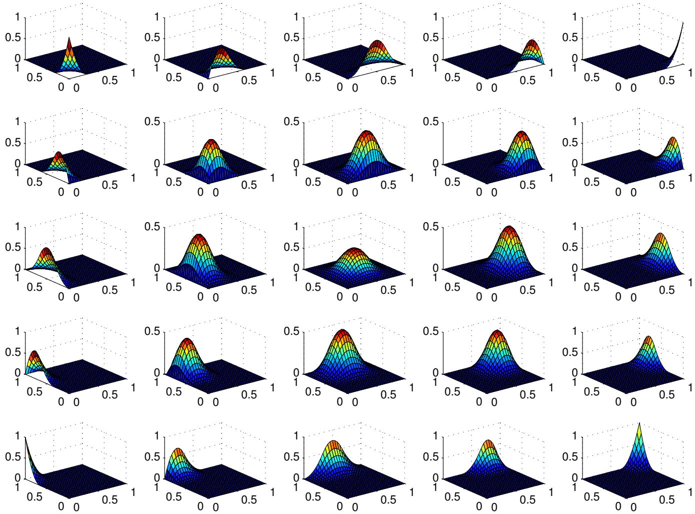
NURBS in $\mathbb{R}^d$ are projections of B-splines in $\mathbb{R}^{d+1}$.

In practice, a weight w_i is associated to each B-spline function, to obtain the **NURBS basis functions** and the control points.

The NURBS curve is determined by: degree, knot vector, control points and weights.

Tensor product surfaces: B-splines

B-splines and NURBS surfaces are defined by **tensor product**.

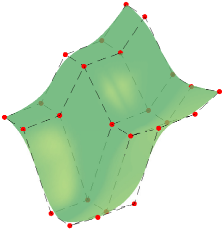


Tensor product surfaces: B-splines

B-splines and NURBS surfaces are defined by **tensor product**.

A control point $\mathbf{C_i} \in \mathbb{R}^d$ is associated to each basis function to define **F**:

$$\mathbf{F}(\zeta) = \sum_{\mathbf{i}} \mathbf{C_i} N_{\mathbf{i},\mathbf{p}}(\zeta)$$



The control points define the **control net**.

With a similar idea, one can define B-spline and NURBS volumes.

The NURBS package

The package is intended to work with NURBS geometries.

- Based on the **NURBS toolbox**, developed by M. Spink in 2000.
- From 2009, extended and maintained by Carlo de Falco and myself.
- Supports curves, surfaces and (simple) volumes.
- Geometry manipulation: rotation, extrusion, revolution...
- It also serves for basis function evaluation (B-splines and NURBS).
- Most of the algorithms come from **The NURBS book**.

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Some technical info:

- The package is part of the octave-forge project.
- All the functions are well documented, including several examples.
- Several tests included, but still some are missing (27 of 68 functions).
- The package contains 11 oct-files.

http://octave.sourceforge.net/nurbs/

NURBS curves: definition in the NURBS package

The construction and manipulation of NURBS geometries is based on a **structure** with the following fields:

- **number**: the number of control points.
- **coefs**: control points coordinates (for NURBS also the weights).
- **order**: the degree plus one.
- **knots**: the knot vector in each direction.

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NURBS geometries can be constructed "by hand" with the function **nrbmak**, giving the **knot vector** and the **control points**.

```
crv = nrbmak(coefs, knt);
```

The package contains several functions to define simple geometries: **nrbline**, **nrbrect**, **nrbcirc**, **nrbcylind**.

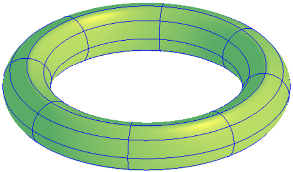
Examples of functions in the package

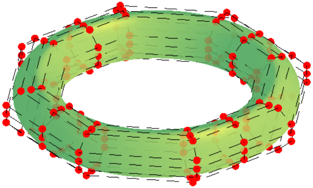
Functions to plot: **nrbkntplot**, **nrbctrlplot**

Revolution and extrusion: **nrbrevolve**, **nrbextrude**

Affine transformations: **nrbtform**

Extract the boundaries of a NURBS: **nrbextract**





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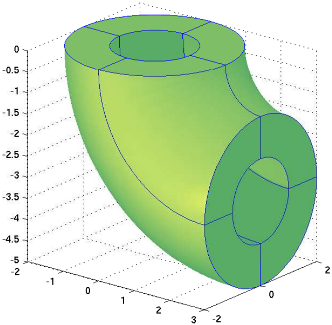

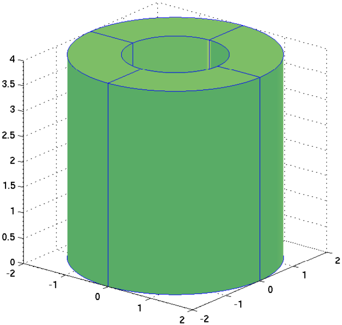

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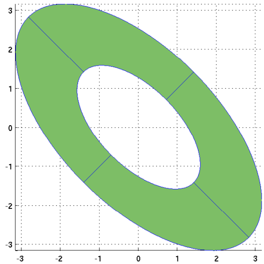

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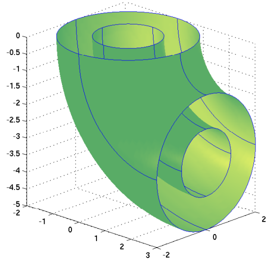

Some doubts and work to do

- Add more tests.
- Compatibility with the **splines package**?
- The package is not intended to be a CAD software. But it would be so nice to move the control points in the figure...

Isogeometric Analysis: overview

Isogeometric Analysis (IGA) is a method for discretization of partial differential equations, similar to the finite element method (FEM).

The idea is to run the simulation directly on a NURBS geometry, approximating the solution also with NURBS (or splines) functions.

It is having a big impact in computational engineering, especially in computational mechanics.

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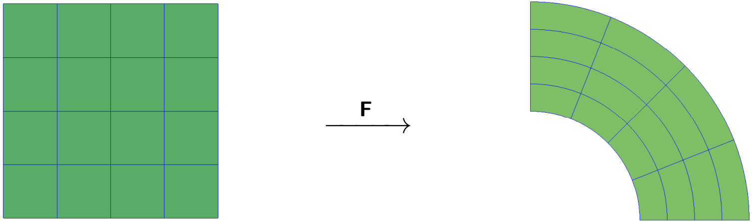
Compared to FEM, the method provides:

- Higher continuity of basis functions.
- Easier mesh generation and refinement.

The concept of IGA

Hughes, Cottrell, Bazilevs (2005)

Reference domain $\widehat{\Omega} = (0,1)^d$, and **physical domain** $\Omega = \mathbf{F}(\widehat{\Omega})$.



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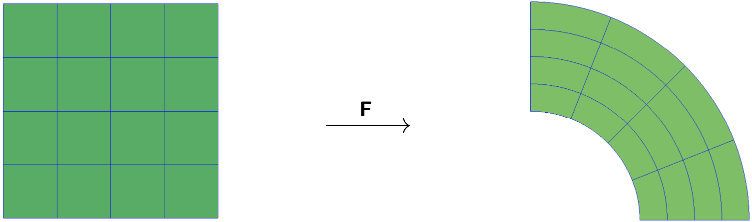

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In $\widehat{\Omega}$ we take the space of NURBS functions, $\widehat{V}_h := \mathrm{span}\{R_{i,p}\}$.

Isoparametric paradigm: in Ω we define $V_h := \mathrm{span}\{v_i := R_{i,p} \circ \mathbf{F}^{-1}\}$.

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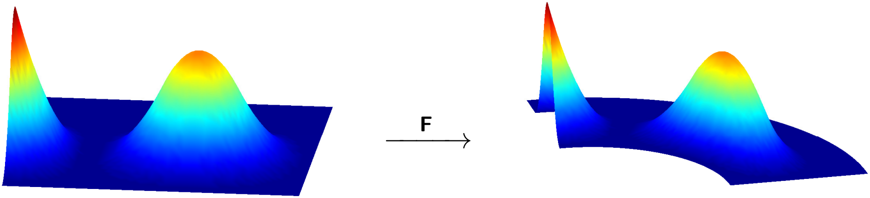

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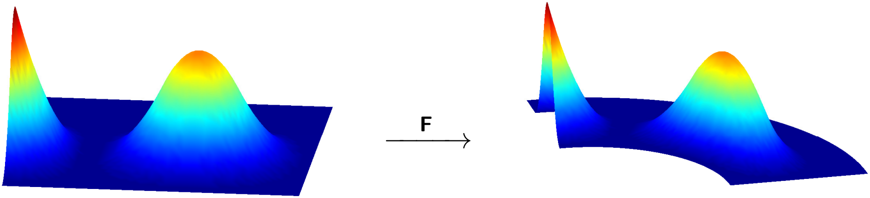

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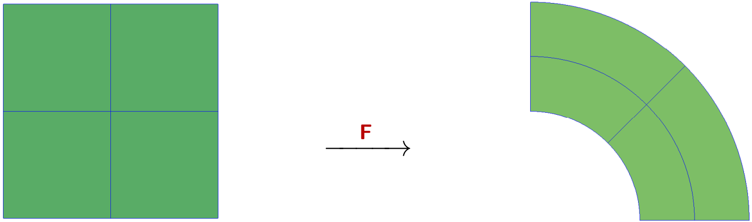
Once the space is defined, everything is similar to FEM: numerical integration, assembly of the global matrices, solution of the linear system...

Refinement in IGA

The coarsest mesh is given by the parametrization **F** of the geometry.

Coarsest mesh: geometry description



$\{R_i\}_{i=1}^{N_0}$ $\{v_i = R_i \circ \mathbf{F}^{-1}\}_{i=1}^{N_0}$

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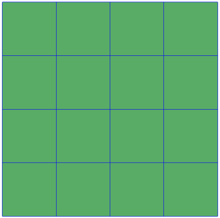
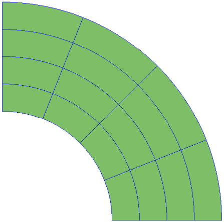
First refinement step



$\xrightarrow{\mathbf{F}}$



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A better approximation is obtained by refining the space, either taking a finer mesh (*h*-refinement) or raising the degree (*p*-refinement).

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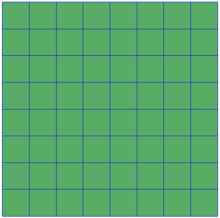
Second refinement step



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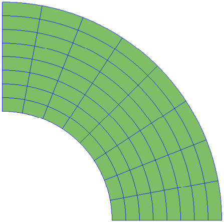

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A better approximation is obtained by refining the space, either taking a finer mesh (*h*-refinement) or raising the degree (*p*-refinement).

The geometry Ω and the parametrization **F** remain **fixed** after refinement.

Since the mesh is structured, refinement is very easy.

GeoPDEs: development

GeoPDEs was originally developed in Pavia, by Carlo de Falco, Alessandro Reali, and myself.

- 2009/2010: join forces to obtain a single/uniform code.
- 2010: first public release of GeoPDEs.
- 2011: presentation at the first IGA conference.
- 2012: version 2.0, efficient use of tensor-product features.
- 2015: version 2.1 (to be released), dimension independent.
- 2016 (expected): adaptivity with hierarchical splines.

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Other developers and contributors

Andrea Bressan, Elena Bulgarello, Durkbin Cho, Jacopo Corno, Adriano Côrtes, Luca Dedè, Sara Frizziero, Eduardo M. Garau, Timo Lähivaara, Marco Pingaro, Anna Tagliabue.

GeoPDEs: description of the software

GeoPDEs consists of a set of interrelated **packages** for different problems:

- **base**: the main package, with examples for Laplace problem.
- **elasticity**: a simple package for linear elasticity problems.
- **fluid**: Stokes' equations, with generalization of face finite elements.
- **maxwell**: Maxwell equations, generalization of edge finite elements.
- **multipatch**: extension to multi-patch defined geometries.

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The **main features** (structures, classes and functions) are defined in **geopdes_base**.

The other packages are based on the structures defined in **base**, with the same nomenclature in each package.

All the packages are built as in octave-forge.

The main structures of GeoPDEs

GeoPDEs has been implemented following an abstract framework.

The code is based on **three** main **structures/classes**:

- **Geometry**: the parametrization **F** and its derivatives.
- **Mesh**: the partition of the domain for **numerical integration**.
- **Space**: the **basis functions** of the approximation space.

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- **Space**: the **basis functions** of the approximation space.

The **geometry** can use the NURBS package, but is not limited to it.

In version 2.0, **mesh** and **space** became classes, to avoid precomputing.

The structures/classes are used in **different applications** without changes.

Other important features

The core of GeoPDEs are the **operator** functions, a family of functions to assemble the matrices and vectors of the method.

This is the most time-consuming part. For efficiency, they are implemented in **oct-files**.

Several examples are already present: Laplacian, bilaplacian, convection terms, SUPG stabilization, Stokes, linear elasticity, Maxwell equations ...

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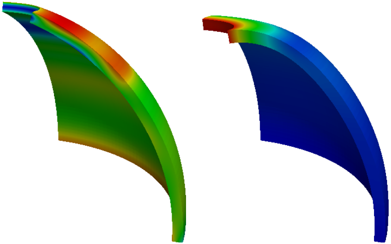
GeoPDEs also includes several functions for postprocessing.

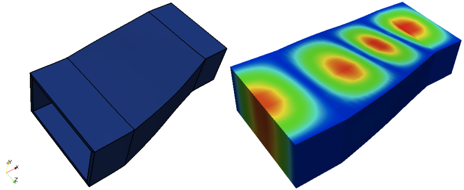
- Export to Paraview.
- Evaluate the solution at given points.
- Compute the error in academic problems with known solution.

Some examples

Linear elasticity.

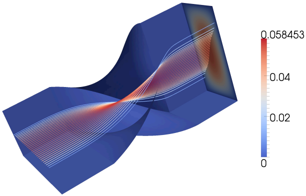


Maxwell problem in a deformed waveguide.



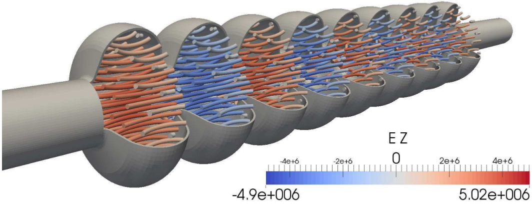
Stokes flow in a twisted pipe.

Bressan, Sangalli (2012)



The TM_{010} mode for a TESLA cavity.

Courtesy of Jacopo Corno



Current and future work

I am currently working on

- A dimension-independent implementation (it will be released soon).
  - Reduction of number of classes, functions and lines of code.
  - Simplifies the imposition of boundary conditions.
- Hierarchical splines with adaptivity (joint work with E. Garau).
  - Still in a preliminary stage.
  - Works in geopdes_base. Needs to be extended to other packages.
- Some theorems that (almost) nobody will understand.

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- Hierarchical splines with adaptivity (joint work with E. Garau).
  - Still in a preliminary stage.
  - Works in geopdes_base. Needs to be extended to other packages.
- Some theorems that (almost) nobody will understand.

Actually, I should also work on

- Adding a test suite. Only 3 functions have a test.
- Change the repository (subversion in sourceforge).
- Rewrite the classes using classdef.
- A new web page.

Current and future work

I am currently working on

- A dimension-independent implementation (it will be released soon).
  - Reduction of number of classes, functions and lines of code.
  - Simplifies the imposition of boundary conditions.
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http://geopdes.sourceforge.net

Thanks for your attention!