The NURBS and GeoPDEs packages
Octave software for research on IGA

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Motivation

In 2008/2009 in Pavia we started to work in isogeometric analysis within the GeoPDEs project. Software development was one of the objectives.

**Starting point:** different codes, different problems, different developers.

**First goal:** a uniform implementation of the different codes.

**Second goal:** it should be clear and easy to use, for didactic purposes, and for new researchers coming into the research group.

The result were two Octave packages: the **NURBS** package, for geometry construction and manipulation, and **GeoPDEs**, for isogeometric methods.
1. **The NURBS package: B-splines and NURBS**
   - B-splines and NURBS: mathematical definitions
   - Functions and examples

2. **The GeoPDEs package: isogeometric analysis**
   - Isogeometric analysis: definition
   - The development of GeoPDEs
   - Some examples
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**Non Uniform Rational B-Splines (NURBS)**

NURBS (non-uniform rational B-splines) are probably the most commonly used CAD technology for engineering design.

NURBS are a generalization of B-splines.
B-splines: definition

Given an ordered knot vector \( \xi_1 \leq \ldots \leq \xi_{n+p+1} \),

define the \( n \) B-splines of degree \( p \) by the recursion formula

\[
N_{i,0}(\zeta) = \begin{cases} 
1 & \text{if } \xi_i \leq \zeta \leq \xi_{i+1} \\
0 & \text{otherwise}
\end{cases}
\]

\[
N_{i,p}(\zeta) = \frac{\zeta - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\zeta) + \frac{\xi_{i+p+1} - \zeta}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\zeta)
\]
B-splines: definition

Given an ordered knot vector $\xi_1 \leq \ldots \leq \xi_{n+p+1}$, define the $n$ B-splines of degree $p$ by the recursion formula

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Degree 2

Degree 3
B-splines: the basis functions

B-spline **basis functions** have the following properties:

- They are non-negative and form a partition of unity.
- Locally linearly independent on each knot span \((\xi_i, \xi_{i+1})\)
- The function \(N_{i,p}\) is supported in the interval \([\xi_i, \xi_{i+p+1}]\).
- Piecewise polynomials of degree \(p\), and regularity at most \(p - 1\).
- The **regularity** at \(\xi_i\) is controlled by the **knot multiplicity**.
A B-spline curve in $\mathbb{R}^d$ is defined as a linear combination of B-splines:

$$F(\zeta) = \sum_{i=1}^{n} C_i N_{i,p}(\zeta)$$

To define the parametrization $F$ we only need:
- The basis functions $N_{i,p}$, given by the knot vector.
- The control points $C_i \in \mathbb{R}^d$. 
**NURBS curves: definition**

**NURBS** are rational B-splines, used to represent conic sections.

NURBS in $\mathbb{R}^d$ are projections of B-splines in $\mathbb{R}^{d+1}$.

In practice, a weight $w_i$ is associated to each B-spline function, to obtain the **NURBS basis functions** and the control points.

The NURBS curve is determined by: degree, knot vector, control points and weights.
Tensor product surfaces: B-splines

B-splines and NURBS surfaces are defined by tensor product.
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A control point $C_i \in \mathbb{R}^d$ is associated to each basis function to define $F$:

$$F(\zeta) = \sum_i C_i N_{i,p}(\zeta)$$

The control points define the control net.

With a similar idea, one can define B-spline and NURBS volumes.
The NURBS package

The package is intended to work with NURBS geometries.

- Based on the **NURBS toolbox**, developed by M. Spink in 2000.
- From 2009, extended and maintained by Carlo de Falco and myself.
- Supports curves, surfaces and (simple) volumes.
- Geometry manipulation: rotation, extrusion, revolution...
- It also serves for basis function evaluation (B-splines and NURBS).
- Most of the algorithms come from **The NURBS book**.
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Some technical info:

- The package is part of the octave-forge project.
- All the functions are well documented, including several examples.
- Several tests included, but still some are missing (27 of 68 functions).
- The package contains 11 oct-files.

The construction and manipulation of NURBS geometries is based on a structure with the following fields:

- **number**: the number of control points.
- **coefs**: control points coordinates (for NURBS also the weights).
- **order**: the degree plus one.
- **knots**: the knot vector in each direction.
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NURBS geometries can be constructed “by hand” with the function `nrbmak`, giving the **knot vector** and the **control points**.

```matlab
crv = nrbmak(coefs, knt);
```

The package contains several functions to define simple geometries: `nrbline`, `nrbrect`, `nrbcirc`, `nrbcylind`. 
Examples of functions in the package

Functions to plot: \texttt{nrbkntplot}, \texttt{nrbctrlplot}

Revolution and extrusion: \texttt{nrbrevolve}, \texttt{nrbextrude}

Affine transformations: \texttt{nrbtform}

Extract the boundaries of a NURBS: \texttt{nrbextract}
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Some doubts and work to do

- Add more tests.
- Compatibility with the **splines package**?
- The package is not intended to be a CAD software. But it would be so nice to move the control points in the figure...
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Isogeometric Analysis (IGA) is a method for discretization of partial differential equations, similar to the finite element method (FEM). The idea is to run the simulation directly on a NURBS geometry, approximating the solution also with NURBS (or splines) functions. It is having a big impact in computational engineering, especially in computational mechanics.
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Compared to FEM, the method provides:

- Higher continuity of basis functions.
- Easier mesh generation and refinement.
The concept of IGA
Hughes, Cottrell, Bazilevs (2005)

Reference domain $\hat{\Omega} = (0, 1)^d$, and physical domain $\Omega = F(\hat{\Omega})$.

We want to solve numerically a certain problem in the NURBS domain $\Omega$.
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We want to solve numerically a certain problem in the NURBS domain $\Omega$. In $\hat{\Omega}$ we take the space of NURBS functions, $\hat{V}_h := \text{span}\{R_{i,p}\}$.

Isoparametric paradigm: in $\Omega$ we define $V_h := \text{span}\{v_i := R_{i,p} \circ F^{-1}\}$. 
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Once the space is defined, everything is similar to FEM: numerical integration, assembly of the global matrices, solution of the linear system...
Refinement in IGA

The coarsest mesh is given by the parametrization $F$ of the geometry.

**Coarsest mesh: geometry description**

\[
\{ R_i \}_{i=1}^{N_0}
\]

\[
\{ v_i = R_i \circ F^{-1} \}_{i=1}^{N_0}
\]

A better approximation is obtained by refining the space, either taking a finer mesh ($h$-refinement) or raising the degree ($p$-refinement).

The geometry $\Omega$ and the parametrization $F$ remain fixed after refinement.

Since the mesh is structured, refinement is very easy.
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First refinement step

\[ \{ R_i \}_{i=1}^{N_1} \]

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A better approximation is obtained by refining the space, either taking a finer mesh ($h$-refinement) or raising the degree ($p$-refinement).
The coarsest mesh is given by the parametrization \( F \) of the geometry.

Second refinement step

\[
\{ R_i \}_{i=1}^{N_2} \rightarrow \{ v_i = R_i \circ F^{-1} \}_{i=1}^{N_2}
\]

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GeoPDEs: development

GeoPDEs was originally developed in Pavia, by Carlo de Falco, Alessandro Reali, and myself.

- 2009/2010: join forces to obtain a single/uniform code.
- 2010: first public release of GeoPDEs.
- 2011: presentation at the first IGA conference.
- 2012: version 2.0, efficient use of tensor-product features.
- 2015: version 2.1 (to be released), dimension independent.
- 2016 (expected): adaptivity with hierarchical splines.
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Other developers and contributors
Andrea Bressan, Elena Bulgarello, Durkbin Cho, Jacopo Corno, Adriano Côrtes, Luca Dedè, Sara Frizziero, Eduardo M. Garau, Timo Lähivaara, Marco Pingaro, Anna Tagliabue.
GeoPDEs: description of the software

GeoPDEs consists of a set of interrelated packages for different problems:

- **base**: the main package, with examples for Laplace problem.
- **elasticity**: a simple package for linear elasticity problems.
- **fluid**: Stokes’ equations, with generalization of face finite elements.
- **maxwell**: Maxwell equations, generalization of edge finite elements.
- **multipatch**: extension to multi-patch defined geometries.
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The main features (structures, classes and functions) are defined in `geopdes_base`.

The other packages are based on the structures defined in `base`, with the same nomenclature in each package.

All the packages are built as in octave-forg.
The main structures of GeoPDEs

GeoPDEs has been implemented following an abstract framework. The code is based on three main structures/classes:

- **Geometry**: the parametrization \( F \) and its derivatives.
- **Mesh**: the partition of the domain for **numerical integration**.
- **Space**: the **basis functions** of the approximation space.
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The geometry can use the NURBS package, but is not limited to it.

In version 2.0, mesh and space became classes, to avoid precomputing.

The structures/classes are used in different applications without changes.
Other important features

The core of GeoPDEs are the operator functions, a family of functions to assemble the matrices and vectors of the method.

This is the most time-consuming part. For efficiency, they are implemented in oct-files.

Several examples are already present: Laplacian, bilaplacian, convection terms, SUPG stabilization, Stokes, linear elasticity, Maxwell equations ...
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GeoPDEs also includes several functions for postprocessing.
- Export to Paraview.
- Evaluate the solution at given points.
- Compute the error in academic problems with known solution.
Some examples

Linear elasticity. Maxwell problem in a deformed waveguide.

Stokes flow in a twisted pipe. The TM$_{010}$ mode for a TESLA cavity.

Bressan, Sangalli (2012)  Courtesy of Jacopo Corno
Current and future work

I am currently working on

- A dimension-independent implementation (it will be released soon).
  - Reduction of number of classes, functions and lines of code.
  - Simplifies the imposition of boundary conditions.
- Hierarchical splines with adaptivity (joint work with E. Garau).
  - Still in a preliminary stage.
  - Works in geopdes_base. Needs to be extended to other packages.
- Some theorems that (almost) nobody will understand.
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- Adding a test suite. Only 3 functions have a test.
- Change the repository (subversion in sourceforge).
- Rewrite the classes using classdef.
- A new web page.
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http://geopdes.sourceforge.net

Thanks for your attention!

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